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Polynomial operators and local smoothness classes on the unit interval

H.N. Mhaskar

Department of Mathematics, California State University, Los Angeles, CA 90032, USA

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Abstract

We obtain a characterization of local Besov spaces of functions on $[-1, 1]$ in terms of algebraic polynomial operators. These operators are constructed using the coefficients in the orthogonal polynomial expansions of the functions involved. The example of Jacobi polynomials is studied in further detail. A by-product of our proofs is an apparently simple proof of the fact that the Cesàro means of a sufficiently high integer order of the Jacobi expansion of a continuous function are uniformly bounded.

Keywords: Polynomial frames; Jacobi polynomials; Local Besov spaces

1. Introduction

It is well known that the polynomials of best approximation to a continuous function on $[-1, 1]$ need not provide a good pointwise approximation. For example, let $f(x) := |x|$, and P_n^* be its best polynomial approximation of degree at most n, $n = 1, 2, \ldots$. Even though f is a piecewise polynomial, the pointwise error $n|f(x) - P_n^*(x)|$ remains bounded away from 0 at a set of points that becomes dense on $[-1, 1]$ as $n \to \infty$ through a subsequence (cf. [1, Theorem 4.1]). Many mathematicians, including Gaier, Ivanov, Saff, and Totik ([6,18], and references therein), have studied the construction of polynomials that provide a near

E-mail address: hmhaska@calstatela.edu.

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best approximation to piecewise analytic functions on the whole interval $[-1, 1]$, and an exponentially fast decaying approximation at points of analyticity of the function.

For example, Gaier [6] constructed a sequence of linear operators $\mathcal{G}_n$ on the space $C[-1, 1]$ of continuous functions on $[-1, 1]$, such that for each $f \in C[-1, 1]$, and integer $n \geqslant 1$, $\mathcal{G}_n(f)$ is a polynomial of degree at most n, and satisfies the following conditions:

$$\max_{x\in[-1,1]} |f(x) - \mathcal{G}_n(f, x)| \leqslant M(f)e^{-\alpha n} + E_{n/6,\infty}(f) \tag{1.1}$$

and if f is regular in the complex neighborhood $|z - x_0| \leqslant d$ of a point $x_0 \in [-1, 1]$, then

$$|f(x_0) - \mathcal{G}_n(f, x_0)| \leqslant M(f)d^{-4} \exp(-cd^2 n), \tag{1.2}$$

where $E_{n/6,\infty}(f)$ is the minimal error of uniform approximation of f by polynomials of degree at most $n/6$ (cf. (2.2) below), $M(f)$ is a positive constant depending only on f, and c, α are absolute positive constants. Gaier's construction is based on the Fourier–Chebyshev coefficients of f. In [11], Prestin and this author constructed a sequence of operators $\mathcal{T}_n$ such that $\max_{x\in I} |\mathcal{T}_n(f, x)|$ tends to zero exponentially fast as $n \to \infty$ if f is analytic on I, while $\max_{x\in I} |\mathcal{T}_n(f, x)|$ is larger than a polynomial in $1/n$ if some derivative of f has a jump discontinuity in I.

The techniques in [6,11] are dependent on complex function theory, and are not applicable for local approximation of functions which are not piecewise analytic. In [13], we have given a construction of operators, similar to those in [11], but applicable to piecewise smooth functions (with a commensurate rate of decay on intervals of smoothness). In this paper, we construct polynomial operators, whose behavior on subintervals of $[-1, 1]$ *characterizes* the local Besov spaces to which the function may belong on these subintervals. These operators are based on the coefficients of an orthogonal polynomial expansion of the function. The periodic analogue of these results is given in [15], where several numerical examples are discussed in detail.

In the next section, we state our main result in a very general setting. This will identify the conditions on the various matrices and measures needed in the construction of our operators. In turn, the construction of these matrices, measures, etc. will be discussed in Section 3 in the context of the Jacobi polynomials. The proofs of the results in Sections 2 and 3 will be presented in Section 4.

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2. The results in a general setting

In this section, we describe our main results in the setting of a general orthogonal polynomial system, identifying the various conditions that the polynomial operators should satisfy. These conditions will then be verified in the context of Jacobi polynomials.

Let μ be a positive, Borel measure on $[-1, 1]$, and $\mathcal{S}_\mu$ denote the support of μ. If $A \subseteq [-1, 1]$ is a Borel set, $\mu(A) > 0$, and $f : A \to \mathbb{R}$ is μ-measurable, we write

$$\|f\|_{A,p} := \|f\|_{\mu;A,p} := \begin{cases} \left\{ \int_A |f(t)|^p d\mu(t) \right\}^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \mu - \operatorname{ess\,sup}_{t\in A} |f(t)| & \text{if } p = \infty. \end{cases} \tag{2.1}$$

The space $L^p(A) := L^p(\mu; A)$ consists of all μ-measurable functions f with $\|f\|_{A,p} < \infty$, with the usual convention that two functions are considered equal if they are equal μ-almost everywhere. The symbol $X^p(A) = X^p(\mu; A)$ will denote the space $L^p(A)$ if $1 \leqslant p < \infty$ and the space of all uniformly continuous, bounded functions on A (equipped with the norm $\|\cdot\|_{A,\infty}$) if $p = \infty$. If $A \subseteq [-1, 1]$ is a closed set, the symbol $C_0^\infty(A)$ will denote the class of infinitely differentiable functions f on $[-1, 1]$, such that $f(x) = 0$ if $x \in [-1, 1] \setminus A$. In the sequel, μ will be a fixed, finite measure, and we will often omit its mention from the notations. Also, if $A = \mathcal{S}_\mu$, we will omit it from the notations; for example, we will write $\|f\|_p := \|f\|_{\mathcal{S}_\mu,p}$. We will assume that $\mathcal{S}_\mu$ is an infinite set.

There are many equivalent ways of defining Besov spaces (cf. [5]). We find it most convenient to define them using the sequence of degrees of approximation of the functions involved. For $x \geqslant 0$, the class of all algebraic polynomials of degree at most x will be denoted by Π_x. For $f \in X^p$ and $x \geqslant 0$, we define the degree of approximation of f from Π_x by

$$E_{x,p}(f) := E_{\mu;x,p}(f) := \min_{P \in \Pi_x} \|f - P\|_p. \tag{2.2}$$

Next, we define a sequence space as follows. Let $0 < \rho \leqslant \infty$, $\gamma > 0$, and $\mathbf{a} = \{a_n\}_{n=0}^\infty$ be a sequence of real numbers. We define

$$\|\mathbf{a}\|_{\rho,\gamma} := \begin{cases} \left\{ \displaystyle\sum_{n=0}^\infty 2^{n\gamma\rho} |a_n|^\rho \right\}^{1/\rho} & \text{if } 0 < \rho < \infty, \\ \displaystyle\sup_{n \geqslant 0} 2^{n\gamma} |a_n| & \text{if } \rho = \infty. \end{cases} \tag{2.3}$$

The space of sequences $\mathbf{a}$ for which $\|\mathbf{a}\|_{\rho,\gamma} < \infty$ will be denoted by $\mathsf{b}_{\rho,\gamma}$. For $1 \leqslant p \leqslant \infty$, $0 < \rho \leqslant \infty$, $\gamma > 0$, the Besov space $B_{p,\rho,\gamma} := B_{\mu;p,\rho,\gamma}$ consists of functions $f \in X^p$ for which the sequence $\{E_{2^n,p}(f)\} \in \mathsf{b}_{\rho,\gamma}$. For $x_0 \in [-1, 1]$, the local Besov space $B_{p,\rho,\gamma}(x_0) := B_{\mu;p,\rho,\gamma}(x_0)$ consists of functions $f \in X^p$ with the following property: There exists an interval I, centered at x_0 such that for every $\phi \in C_0^\infty(I)$, the function $f\phi \in B_{p,\rho,\gamma}$. This interval may depend upon f and x_0 in addition to the other parameters.

Our objective in this paper is to characterize local Besov spaces in terms of operators based on the coefficients of the target function in terms of an orthogonal polynomial expansion. We recall [19] that there is a unique system of polynomials $p_n := p_n(\mu) \in \Pi_n$, $n = 0, 1, \ldots$, each p_n having a positive leading coefficient, such that

$$\int p_n p_m \, d\mu = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{2.4}$$

If $f \in X^1$, we define its orthogonal polynomial coefficients by

$$\hat{f}(m) := \hat{f}(\mu; m) := \int f(t) p_m(t) \, d\mu(t), \qquad m = 0, 1, \ldots . \tag{2.5}$$

Our operators will be defined using a bi-infinite matrix. If $H = (h_{j,n})_{\substack{j=0,1,\ldots \\ n=1,2,\ldots}}$ is a bi-infinite matrix such that for each $n \geqslant 1$, $h_{j,n} = 0$ if j is greater than some integer, we will define the operator

$$\sigma_n(H, f, x) := \sigma_n(\mu; H, f, x) := \sum_{j=0}^{\infty} h_{j,n}\hat{f}(j)p_j(x), \qquad f \in X^1. \tag{2.6}$$

We note that $\sigma_n(H, f)$ is a polynomial for each $n \geqslant 1$, and with

$$\Phi_n(H, x, y) := \Phi_n(\mu; H, x, y) := \sum_{j=0}^{\infty} h_{j,n}p_j(x)p_j(y), \qquad x, y \in \mathbb{R}, \tag{2.7}$$

we have the representation

$$\sigma_n(H, f, x) = \int f(y)\Phi_n(H, x, y)d\mu(y), \qquad f \in X^1,\ x \in \mathbb{R}. \tag{2.8}$$

In the sequel, we find it convenient to define $h_{k,t} := 0$ for any real $x < t$. Correspondingly, we also define $\sigma_t(H, f) := 0$ and $\Phi_t(H, x, y) := 0$ for all real $t < 1$. For $n \geqslant 0$, we write

$$\tau_n(H, f) := \tau_n(\mu; H, f) := \sigma_{2^n}(H, f) - \sigma_{2^{n-1}}(H, f). \tag{2.9}$$

We note that if $h_{j,n} = 0$ for $j > n$, $n = 0, 1, \ldots$, our notation implies that $\sigma_n(H, f) \in \Pi_n$, and $\tau_n(H, f) \in \Pi_{2^n}$.

For $Q \geqslant 1$, the set $\mathcal{S}^Q := \mathcal{S}^Q(\mu)$ consists of all matrices H such that $h_{j,n} = 0$ if $j > n$, $h_{j,n} = 1$ if $0 \leqslant j \leqslant n/2$,

$$\sup_{n \geqslant 1, x \in \mathcal{S}_\mu} \|\Phi_n(H, x, \cdot)\|_1 < \infty \tag{2.10}$$

and for every $x_0 \in \mathcal{S}_\mu$ and $\eta > 0$, there exists a constant $c = c(x_0, \eta)$ such that,

$$\sup_{n \geqslant 1,\ y \in \mathcal{S}_\mu \setminus [x_0 - \eta, x_0 + \eta]} n^Q |\Phi_n(H, x, y)| < c, \qquad |x - x_0| \leqslant \eta/2. \tag{2.11}$$

An example of matrices in $\mathcal{S}^Q$ is given in Theorem 3.1 in the next section.

In the sequel, we adopt the following convention regarding constants. The symbols $c, c_1, \ldots$ will denote positive constants depending upon μ, ρ, γ, p, and Q, in addition to any explicitly mentioned quantities. Their value may be different at different occurences, even within the same formula.

We will characterize the local Besov spaces using the norms of the operators $\tau_n(H, f)$ on subintervals of $[-1, 1]$. We would also like to give a characterization using values of these polynomials at certain points. As expected, this depends upon a quadrature formula, and a connection between discrete and continuous norms of a polynomial. Accordingly, we introduce some further notation. If ν is a signed, Borel measure on $[-1, 1]$, its total variation measure will be denoted by $|\nu|$ (or $|d\nu|$ in the context of integration). For a ν-measurable function f, and ν-measurable subset $A \subseteq [-1, 1]$, we write $\|f\|_{\nu; A, p} := \|f\|_{|\nu|; A, p}$. As before, we will omit the mention of the set A if $A = [-1, 1]$.

The measure ν will be called an M–Z quadrature measure of order n (for μ) if its support is a subset of the support of μ,

$$\|P\|_{\nu; p} \leqslant c\|P\|_{\mu; p}, \qquad P \in \Pi_n,\ 1 \leqslant p \leqslant \infty \tag{2.12}$$

and

$$\int P\,d\nu = \int P\,d\mu, \qquad P \in \Pi_n. \tag{2.13}$$

For a sequence $\{\nu_n\}$ of M–Z quadrature measures, it is assumed tacithy that the constant c in (2.12) is independent of n. An estimate of form (2.12) is often called a Marcinkiewicz–Zygmund-type inequality. Many examples of such estimates are known in the literature (for example, [8,12], and references therein). In the next section, we will mention an example in the case of Jacobi polynomials.

For the purpose of future reference, we note here that if ν_n is an M–Z quadrature measure of order n, $n = 0, 1, \ldots$, H is a bi-infinite matrix with $h_{j,n} = 0$ for all $j > n$, and (2.10) holds, then also the following estimate holds.

$$\sup_{x\in\mathcal{S}_\mu} \|\Phi_m(H, x, \cdot)\|_{\nu_n;1} < c, \qquad 0 \leqslant m \leqslant n, \; n = 0, 1, \ldots\,. \tag{2.14}$$

Our main theorem in this paper is the following.

Theorem 2.1. *Let* $1 \leqslant p \leqslant \infty$, $f \in X^p$, $x_0 \in [-1, 1]$, $0 < \rho \leqslant \infty$, $\gamma > 0$, $Q > \max(1, \gamma)$, $H \in \mathcal{S}^Q$, *and for each integer* $n \geqslant 0$, *let* ν_n *be an M–Z quadrature measure of order* $6(2^n)$. *Then*

$$f = \sum_{n=0}^{\infty} \tau_n(H, f) = \sum_{n=0}^{\infty} \int \tau_n(H, f, t)\{\Phi_{2^{n+1}}(H, \cdot, t) - \Phi_{2^{n-2}}(H, \cdot, t)\}d\nu_n(t), \tag{2.15}$$

with the series converging in the sense of X^p. *Moreover, the following are equivalent.*

(a) $f \in B_{p,\rho,\gamma}(x_0)$.
(b) *There exists an interval I, centered at* x_0, *such that for every* $\phi \in C_0^\infty(I)$, $\{\|\tau_n(H, f\phi)\|_p\} \in \mathsf{b}_{\rho,\gamma}$.
(c) *There exists an interval I, centered at* x_0, *such that for every* $\phi \in C_0^\infty(I)$, $\{\|\tau_n(H, f\phi)\|_{\nu_n;p}\} \in \mathsf{b}_{\rho,\gamma}$.
(d) *There exists an interval I, centered at* x_0, *such that* $\{\|\tau_n(H, f)\|_{I,p}\} \in \mathsf{b}_{\rho,\gamma}$.
(e) *There exists an interval I, centered at* x_0, *such that* $\{\|\tau_n(H, f)\|_{\nu_n;I,p}\} \in \mathsf{b}_{\rho,\gamma}$.

In all the anticipated applications, the measures ν_n will be supported on finite sets $\mathcal{C}_n$ of points in $[-1, 1]$. In this case, (2.15) presents $\{\tau_n(H, f, t)\}_{t\in\mathcal{C}_n}$ as the sequence of coefficients in a series representation of *f*, and the equivalence between parts (a) and (e) shows that the local Besov spaces can be characterized using the absolute values of these coefficients. We note here that the operators are defined using global information about the function, in the form of the coefficients $\hat{f}(k)$, and yet, their behavior is different near different points, depending upon the smoothness of *f* near these points. Moreover, the local Besov spaces are characterised in terms of the norms of $\{\tau_n(H, f)\}$ themselves, rather than their approximation to *f*, as in (1.1), (1.2). Theorem 3.1 below can be used to construct (in the case of Jacobi polynomials) matrices that belong to $\mathcal{S}^Q$ for every integer *Q*. Therefore, a single sequence of operators may be used for the characterization of all the smoothness classes, a situation expected in polynomial approximation. The equivalence between (a) and (b) (or (c)) shows that the apparently global condition that $f \in X^p$ is really not necessary if one is interested only in the behavior of *f* near a point. Working with the coefficients $\widehat{f\phi}$ in place of those of *f*, the local smoothness conditions imply the global behavior of the operator $\tau_n(H, f\phi)$.

3. Jacobi polynomials

In this section, we illustrate the technical conditions which we discussed in the previous section. Thus, we demonstrate a general construction of the matrix $H \in \mathcal{S}^Q$ in the case of the Jacobi polynomials. We will recall a construction of M–Z quadrature formulas in this case. We will also make an additional observation regarding expansion (2.15).

We recall that the Jacobi weight is defined for $\alpha, \beta > -1$ by

$$w_{\alpha,\beta}(x) := \begin{cases} (1-x)^\alpha(1+x)^\beta & \text{if } x \in (-1,1), \\ 0 & \text{if } x \in \mathbb{R} \setminus (-1,1). \end{cases}$$

The corresponding measure $\mu_{\alpha,\beta}$ is defined by $d\mu_{\alpha,\beta}(x) := w_{\alpha,\beta}(x)dx$, and we will simplify our notations by writing α, β in place of μ; for example, we write $\|f\|_{\alpha,\beta;A,p}$ instead of $\|f\|_{\mu_{\alpha,\beta};A,p}$. We recall the definition of the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}$ [19]. For integer $n \geqslant 0$, $P_n^{(\alpha,\beta)} \in \Pi_n$ has a positive leading coefficient, and with

$$\kappa_n^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, \tag{3.1}$$

we have for integers $n, m \geqslant 0$,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)dx = \begin{cases} \kappa_n^{(\alpha,\beta)} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{3.2}$$

Thus, $p_n(\mu_{\alpha,\beta}) = {\kappa_n^{(\alpha,\beta)}}^{-1/2} P_n^{(\alpha,\beta)}$.

3.1. M–Z quadrature

Nevai [17, Theorem 25, p. 168] has given an example of M–Z quadratures for the Jacobi weights. For $m \geqslant 1$, let $\{x_{k,m}\}_{k=1}^m$ be the zeros of $P_m^{(\alpha,\beta)}$, and

$$\lambda_{k,m} := \left\{ \sum_{j=0}^{m-1} {\kappa_j^{(\alpha,\beta)}}^{-1} P_j^{(\alpha,\beta)}(x_{k,m})^2 \right\}^{-1}, \qquad k = 1, \ldots, m.$$

Nevai has proved that for $m \geqslant cn$, the measure ν_m^* that associates the mass $\lambda_{k,m}$ with each $x_{k,m}$ is an M–Z quadrature measure of order *n*. It is possible to construct M–Z quadrature measures supported at an “arbitrary” system of points, subject to certain denseness conditions. We plan to address this question in another paper.

3.2. The matrices and Cesàro means

The following theorem gives a general construction for matrices in $\mathcal{S}^Q(\alpha, \beta)$.

Theorem 3.1. *Let $\alpha, \beta \geqslant -1/2$, $\delta > 0$, $Q \geqslant 0$, $K \geqslant Q + \alpha + \beta + 2$ be an integer, and $h : [0, \infty) \to \mathbb{R}$ be a function which is a K times iterated integral of a function of bounded variation, $h'(x) = 0$ if $0 \leqslant x \leqslant \delta$, and $h(x) = 0$ if $x > c$. Then the matrix $H = (h_{k,n})$ defined by $h_{k,n} = h(k/n)$, $n \geqslant 1$, satisfies (2.10) and (2.11) with $\mu_{\alpha,\beta}$ in place of μ. In particular, if $h(x) = 1$ for $0 \leqslant x \leqslant 1/2$ and $h(x) = 0$ for $x > 1$, then $H \in \mathcal{S}^Q(\alpha, \beta)$.*

We recall that if $k > -1$, and

$$C_n^{[k]}(\alpha, \beta; x, y) := \sum_{v=0}^{n} \binom{n - v + k}{k} {\kappa_v}^{(\alpha,\beta)^{-1}} P_v^{(\alpha,\beta)}(x) P_v^{(\alpha,\beta)}(y), \tag{3.3}$$

the Cesàro means of order k of $f \in X^1$ are defined by

$$S_n^{[k]}(\alpha, \beta; f, x) := \binom{n + k}{k}^{-1} \int_{-1}^{1} f(y) C_n^{[k]}(\alpha, \beta; x, y) w_{\alpha,\beta}(y) dy. \tag{3.4}$$

The following theorem is well known [2,19].

Theorem 3.2. *Let $\alpha, \beta \geqslant -1/2$, $k > \max(\alpha, \beta) + 1/2$. Then for $n = 1, 2, \ldots$,*

$$\max_{x \in [-1,1]} \|C_n^{[k]}(\alpha, \beta; x, 1)\|_1 \leqslant c n^k. \tag{3.5}$$

In Theorem 3.1, the fact that H satisfies (2.10) can be obtained using Theorem 3.2 by a simple summation by parts argument as in [10]. However, the bounds for the decay of the Cesàro kernels $C_n^{[k]}$, similar to (2.11) and known to this author [4], do not improve with the order k. Our method to prove such bounds for the kernels does not use the properties of the Cesàro means. Instead, Theorem 3.2 follows in the case of integer k as an application of Lemma 4.6 obtained during our proof of (2.10). We feel that this proof is simpler than that given in [19].

3.3. Series expansion

Finally, we make a remark about representation (2.15). The system of functions $\{\Phi_{2^{n+1}}(H, \cdot, t) - \Phi_{2^{n-2}}(H, \cdot, t)\}_{t \in \mathcal{C}_n}$ is not linearly independent, and hence, the coefficients of the series representation in (2.15) are not uniquely determined. The following theorem shows that in the case when H is as in Theorem 3.1, the behavior of an arbitrary coefficient sequence which works in (2.15) implies the local Besov conditions.

Theorem 3.3. *Let $\alpha, \beta \geqslant -1/2$, $\mu = \mu_{\alpha,\beta}$, $1 \leqslant p \leqslant \infty$, $f \in X^p$, $x_0 \in [-1, 1]$, $0 < \rho \leqslant \infty$, $\gamma > 0$, $Q > \max(\gamma, 1)$, $H \in \mathcal{S}^Q(\alpha, \beta)$ be as in Theorem 3.1, and for each integer $n \geqslant 0$,* *let ν_n be an M–Z quadrature measure of order* $6(2^n)$. *For* $n \geqslant 0$, *let* d_n *be a* ν_n *measurable function, and* $\|d_n\|_{\nu_n;p} \leqslant c$. *Suppose that*

$$f = \sum_{n=0}^{\infty} \int d_n(t)\{\Phi_{2^{n+1}}(\alpha, \beta; H, \cdot, t) - \Phi_{2^{n-2}}(\alpha, \beta; H, \cdot, t)\} d\nu_n(t), \tag{3.6}$$

where the series converges in the sense of X^p. *If there exists an interval I centered at* x_0 *such that* $\{\|d_n\|_{\nu_n;I,p}\} \in \mathsf{b}_{\rho,\gamma}$, *then* $f \in B_{\alpha,\beta;p,\rho,\gamma}(x_0)$.

We note that since $\{\Phi_{2^{n+1}}(\alpha, \beta; H, \cdot, t) - \Phi_{2^{n-2}}(\alpha, \beta; H, \cdot, t)\}$ are not linearly independent, the converse of Theorem 3.3 cannot hold.

4. Proofs

In order to prove Theorem 2.1, we need some lemmas. The first lemma is a simple consequence of the Riesz–Thorin interpolation theorem [3, Theorem 1.1.1], and we state it in order to refer to it in a convenient way.

Lemma 4.1. *Let* m_1, m_2 *be signed measures* (*having bounded variation*) *on a measure space S, supported on* S_1 *and* S_2, *respectively,* $\Psi : S \times S \to \mathbb{R}$ *be a bounded,* $|m_1| \times |m_2|$ *measurable function,* $\Psi(x, t) = \Psi(t, x)$ *for* $x, t \in S$, $1 \leqslant p \leqslant \infty$, $f \in L^p(|m_1|)$, *and let*

$$T_f(x) := \int f(t)\Psi(x, t) dm_1(t).$$

Then with

$$A = \max\left(\sup_{x \in S_1} \|\Psi(x, \cdot)\|_{|m_2|;1}, \sup_{x \in S_2} \|\Psi(x, \cdot)\|_{|m_1|;1}\right),$$

we have

$$\|T_f\|_{|m_2|;p} \leqslant A\|f\|_{|m_1|;p}. \tag{4.1}$$

Proof. We observe that

$$\begin{aligned} \int |T_f(x)||dm_2(x)| &\leqslant \int\int |f(t)||\Psi(x, t)||dm_1(t)||dm_2(x)| \\ &\leqslant \sup_{t \in S_1} \|\Psi(\cdot, t)\|_{|m_2|;1} \|f\|_{|m_1|;1} \leqslant A\|f\|_{|m_1|;1}. \end{aligned}$$

This proves (4.1) in the case $p = 1$. The case $p = \infty$ is obvious, and the general case follows from the Riesz–Thorin interpolation theorem applied to the linear operator $f \to T_f$. □

The first application of this lemma is the following lemma, summarizing some properties of the operators $\Phi_n(H)$.

Lemma 4.2. *Let* $1 \leqslant p \leqslant \infty$, $f \in X^p$, $x_0 \in [-1,1]$, $0 < \rho \leqslant \infty$, $\gamma > 0$, $Q > \max(1,\gamma)$, $H \in \mathcal{S}^Q$, *and for each integer* $n \geqslant 1$, let v_n *be an M–Z quadrature measure of order* $6(2^n)$. *Then* (2.15) *holds with convergence in the sense of* X^p. *Moreover,*

$$\|\sigma_m(H, f)\|_p \leqslant c\|f\|_p, \qquad m = 0, 1, \ldots . \tag{4.2}$$

Consequently, for $m \geqslant 0$,

$$E_{m,p}(f) \leqslant \|f - \sigma_m(H, f)\|_p \leqslant cE_{m/2,p}(f). \tag{4.3}$$

Proof. Estimate (4.2) follows immediately from (2.10) and Lemma 4.1, applied with $m_2 = m_1 = \mu$. The first estimate of (4.3) is obvious. Since $h_{k,m} = 1$ for $k \leqslant m/2$, we have $\sigma_m(H, P) = P$ for all $P \in \Pi_{m/2}$. Therefore, choosing $P \in \Pi_{m/2}$ with $\|f - P\|_p \leqslant 2E_{m/2,p}(f)$, we obtain

$$\begin{aligned}\|f - \sigma_m(H, f)\|_p &= \|f - P - \sigma_m(H, f - P)\|_p \\ &\leqslant \|f - P\|_p + \|\sigma_m(H, f - P)\|_p \\ &\leqslant c\|f - P\|_p \leqslant cE_{m/2,p}(f).\end{aligned}$$

This proves the second inequality in (4.3). The first equation in (2.15) follows from (4.3) and the definition of τ_n. Next, we observe that $R_x := \Phi_{2^{n+1}}(H, x, \cdot) - \Phi_{2^{n-2}}(H, x, \cdot) \in \Pi_{2^{n+1}}$ and $h_{k,2^{n+1}} - h_{k,2^{n-2}} = 1$ if $2^{n-2} < k \leqslant 2^n$. Since $\widehat{\tau_n(H, f)}(k) \neq 0$ only when $2^{n-2} < k \leqslant 2^n$, and $R_x\tau_n(H, f) \in \Pi_{6(2^n)}$, we see from (2.13) that

$$\tau_n(H, f, x) = \int \tau_n(H, f, t)R(t)d\mu(t) = \int \tau_n(H, f, t)R_x(t)dv_n(t), \qquad x \in \mathbb{R}.$$

This gives the second equation in (2.15). $\square$

Another application of Lemma 4.1 is the following lemma, relating the continuous and discrete norms of polynomials on $[-1, 1]$, as well as on subintervals of $[-1, 1]$.

Lemma 4.3. *Let* $m \geqslant 0$, v *be an M–Z quadrature measure of order* $6m$, *and suppose that there exists a matrix* $H \in \mathcal{S}^Q$ *for some* $Q \geqslant 0$. *Then for* $P \in \Pi_{2m}$,

$$\|P\|_{\mu;p} \leqslant c\|P\|_{v;p} \leqslant c_1\|P\|_{\mu;p}. \tag{4.4}$$

If $J \subset I \subseteq [-1, 1]$ *are intervals, then for* $P \in \Pi_m$,

$$\begin{aligned}\|P\|_{\mu;J,p} &\leqslant c(I, J)\{\|P\|_{v;I,p} + m^{-Q}\|P\|_{v;p}\}, \\ \|P\|_{v;J,p} &\leqslant c(I, J)\{\|P\|_{\mu;I,p} + m^{-Q}\|P\|_{\mu;p}\}.\end{aligned} \tag{4.5}$$

Proof. Let $P \in \Pi_{2m}$. In view of (2.13) and the fact that $h_{k,4m} = 1$ for $0 \leqslant k \leqslant 2m$, we have

$$P(x) = \int P(t)\Phi_{4m}(H, x, t)d\mu(t) = \int P(t)\Phi_{4m}(H, x, t)dv(t).$$

We use Lemma 4.1 with $m_2 = \mu$ and $m_1 = \nu$, and use (2.10), (2.14) for H, to arrive at the first inequality in (4.4). The second inequality is (2.12).

Next, let $P \in \Pi_m$, and $\phi \in C_0^\infty(I)$ be chosen so that $\phi(x) = 1$ if $x \in J$. By the direct theorem of approximation theory [5, Theorem 6.2, Chapter 7], there exists $R \in \Pi_m$ such that

$$\|\phi - R\|_\infty \leqslant c(I, J) m^{-Q}.$$

Therefore, using (4.4) for the polynomial $PR \in \Pi_{2m}$,

$$\begin{aligned}\|P\|_{\mu;J,p} &= \|P\phi\|_{\mu;J,p} \leqslant \|PR\|_{\mu;p} + \|P(\phi - R)\|_{\mu;p}\\ &\leqslant c(I, J)\{\|PR\|_{\nu;p} + m^{-Q}\|P\|_{\mu;p}\}\\ &\leqslant c(I, J)\{\|P\phi\|_{\nu;p} + m^{-Q}\|P\|_{\nu;p}\}\\ &\leqslant c(I, J)\{\|P\|_{\nu;I,p} + m^{-Q}\|P\|_{\nu;p}\}.\end{aligned}$$

This proves the first inequality in (4.5). The second inequality is proved in a similar way. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. First, we prove the equivalence of parts (a)–(c). Let (a) hold, and ϕ be a C^∞ function such that $f\phi \in B_{p,\rho,\gamma}$. In view of (4.3),

$$\begin{aligned}\|\tau_n(H, f\phi)\|_p &\leqslant \|\sigma_{2^n}(H, f\phi) - f\phi\|_p + \|\sigma_{2^{n-1}}(H, f\phi) - f\phi\|_p\\ &\leqslant cE_{2^{n-2},p}(f\phi).\end{aligned}$$

This implies part (b). Conversely, let (b) hold, and ϕ be a C^∞ function as in that part. In view of (2.15),

$$E_{2^n,p}(f\phi) \leqslant \left\|f\phi - \sum_{m=0}^{n} \tau_m(H, f\phi)\right\|_p \leqslant \sum_{m=n+1}^{\infty} \|\tau_m(H, f\phi)\|_p.$$

Since $\{\|\tau_m(H, f\phi)\|_p\} \in \mathsf{b}_{\rho,\gamma}$, the discrete Hardy inequality [5, Lemma 3.4, p. 27] now leads to part (a). The equivalence between parts (b) and (c) is immediate from (4.4).

Next, we will show that part (b) implies part (d), and part (d) implies part (a). Let I be as in part (b), and J (respectively, J_1) be the interval centered at x_0 and length $|I|/2$ (respectively, $|I|/4$). Let $\psi \in C_0^\infty(I)$ be chosen so that $\psi(x) = 1$ for $x \in J$ and $\|\psi\|_\infty = 1$. For $x \in J_1$, we have from (2.11) that for any integer $m \geqslant 1$,

$$\begin{aligned}&\left|\int f(t)(1 - \psi(t))\Phi_m(H, x, t)d\mu(t)\right|\\ &\quad= \left|\int_{\mathcal{S}_\mu \setminus J} f(t)(1 - \psi(t))\Phi_m(H, x, t)d\mu(t)\right|\\ &\quad\leqslant \int_{|x-t| \geqslant |I|/8} |f(t)(1 - \psi(t))\Phi_m(H, x, t)|d\mu(t) \leqslant c(I)m^{-Q} \int |f(t)|d\mu(t)\\ &\quad\leqslant c(I)m^{-Q}\|f\|_p.\end{aligned}$$

Applying this inequality once with $m = 2^n$ and once with $m = 2^{n-1}$, we deduce that

$$\|\tau_n(H, (1-\psi)f)\|_{J_1,\infty} \leqslant c(f, I)2^{-nQ}.$$

Therefore,

$$\begin{aligned}\|\tau_n(H, f)\|_{J_1,p} &\leqslant \|\tau_n(H, \psi f)\|_p + c\|\tau_n(H, (1-\psi)f)\|_{J_1,\infty} \\ &\leqslant \|\tau_n(H, \psi f)\|_p + c(f, I)2^{-nQ}.\end{aligned}$$

Since both the sequences $\{\|\tau_n(H, \psi f)\|_p\}$ and $\{2^{-nQ}\}$ are in $\mathrm{b}_{\rho,\gamma}$, part (d) is proved.

Next, let part (d) hold, I be the interval as in that part, and $\phi \in C_0^\infty(I)$. By the direct theorem of approximation theory [5, Theorem 6.2, Chapter 7], there exists $R \in \Pi_{2^n}$ such that

$$\|\phi - R\|_{[-1,1],\infty} \leqslant c2^{-nQ}.$$

Therefore, using (4.2) and (2.15), we obtain

$$\begin{aligned}E_{2^{n+1},p}(f\phi) &\leqslant \|f\phi - R\sigma_{2^n}(H, f)\|_p \leqslant \|(f - \sigma_{2^n}(H, f))\phi\|_p \\ &\quad + \|(\phi - R)\sigma_{2^n}(H, f)\|_p \\ &\leqslant c(I, \phi, f)\{\|f - \sigma_{2^n}(H, f)\|_{I,p} + 2^{-nQ}\} \\ &\leqslant c(I, \phi, f)\left\{\sum_{m=n+1}^{\infty} \|\tau_m(H, f)\|_{I,p} + 2^{-nQ}\right\}.\end{aligned}$$

The discrete Hardy inequality [5, Lemma 3.4, p. 27] now shows that $f\phi \in B_{p,\rho,\gamma}$. Thus, part (d) implies part (a).

Thus, parts (a)–(d) are equivalent. The equivalence between parts (d) and (e) is a simple consequence of (4.5). □

In the sequel, we will assume that the measure μ is the Jacobi distribution $\mu_{\alpha,\beta}$ and often omit α, β from the notations, when it is not expected to cause confusion.

The proof of Theorem 3.1 requires several technical estimates. We observe first that ${P_n}^{(\alpha,\beta)}(x) = (-1)^n {P_n}^{(\beta,\alpha)}(-x)$, $n = 0, 1, \ldots$. Hence, there is no loss of generality in assuming that $\alpha \geqslant \beta \geqslant -1/2$. The idea behind the proof of (2.10) is the following. We will use the formula (cf. [19, Formulas (4.5.3), (4.3.3), (4.1.1)])

$$\begin{aligned}K_{n,1}(\alpha, \beta; 1, x) &:= \sum_{m=0}^{n} {\kappa_m}^{(\alpha,\beta)^{-1}} {P_m}^{(\alpha,\beta)}(x){P_m}^{(\alpha,\beta)}(1) \\ &= 2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}{P_n}^{(\alpha+1,\beta)}(x) \\ &= \frac{2(\alpha+1)}{2n+\alpha+\beta+2}{\kappa_n}^{(\alpha+1,\beta)^{-1}}{P_n}^{(\alpha+1,\beta)}(x){P_n}^{(\alpha+1,\beta)}(1) \qquad (4.6)\end{aligned}$$

repeatedly along with a summation by parts to obtain an alternate formula for the kernel $\Phi_n(H, 1, x)$. It will then be clear that $\|\Phi_n(H, 1, \cdot)\|_1 \leqslant c$. In view of the convolution structure on the Jacobi polynomials [2], this will lead to (2.10). The estimate required in (2.11) is proved in the case when $x \in (-1, 1)$ and $y \in (-c_1(x), c_2(x))$ using the asymptotic formulas for the Jacobi polynomials and some ideas from [13,14]. The case when $y \notin (-c_1(x), c_2(x))$ requires a more elaborate analysis, involving a product formula proved in [7] by Koornwinder. The details will be organized in a series of lemmas, starting with a general formula for summation by parts.

During the remainder of the proof of Theorem 3.1, we will adopt the following notation. The forward difference operator is defined by

$$\Delta a_\nu = \Delta^1 a_\nu = a_{\nu+1} - a_\nu, \ \Delta^k a_\nu = \Delta(\Delta^{k-1} a_\nu), \ \nu \geqslant 0, \ k \geqslant 2. \tag{4.7}$$

We will write for $x \in \mathbb{R}$,

$$\ell_k(x) = \frac{2x + \alpha + \beta + k}{2(\alpha + k - 1)}, \qquad k \geqslant 2$$

and define a modified difference of a sequence by

$$a_\nu^{[1]} = \Delta a_\nu, \ a_\nu^{[k]} = \frac{a_{\nu+1}^{[k-1]}}{\ell_k(\nu+1)} - \frac{a_\nu^{[k-1]}}{\ell_k(\nu)}, \ \nu \geqslant 0, \ k \geqslant 2. \tag{4.8}$$

The partial summation operators corresponding to these differences are defined by

$$s_m^{[1]} = \sum_{\nu=0}^{m} a_\nu, \ s_m^{[k]} = \sum_{\nu=0}^{m} \ell_k(\nu) s_\nu^{[k-1]}, \ m \geqslant 0, \ k \geqslant 2. \tag{4.9}$$

We define $s_m^{[k]} = 0$ if $m < 0$.

The following lemma describes some properties of the partial summation.

Lemma 4.4. *Let $\{h_\nu\}$ be a sequence with $h_\nu = 0$ if ν is greater than some positive integer, $\{a_\nu\}$ be any sequence. We have for $k = 1, 2, \ldots$,*

$$\sum_{\nu=0}^{\infty} h_\nu a_\nu = (-1)^k \sum_{\nu=0}^{\infty} h_\nu^{[k]} s_\nu^{[k]} \tag{4.10}$$

and for $\nu = 0, 1, \ldots, k = 1, 2, \ldots$,

$$|h_\nu^{[k]}| \leqslant c \sum_{m=0}^{k-1} \left| \frac{\Delta^{k-m} h_\nu}{(\nu+1)^{k+m-1}} \right|. \tag{4.11}$$

Proof. Using the fact that $s_{-1}^{[1]} = 0$, and the fact that $h_\nu = 0$ for sufficiently large ν, a summation by parts shows that

$$\sum_{\nu=0}^{\infty} h_\nu a_\nu = \sum_{\nu=0}^{\infty} h_\nu (s_\nu^{[1]} - s_{\nu-1}^{[1]}) = \sum_{\nu=0}^{\infty} h_\nu s_\nu^{[1]} - \sum_{\nu=0}^{\infty} h_{\nu+1} s_\nu^{[1]} = -\sum_{\nu=0}^{\infty} \Delta h_\nu s_\nu^{[1]}.$$

Hence, (4.10) holds when $k = 1$. Suppose $k \geqslant 2$ and the formula holds for $k - 1$ in place of k. Using the fact that $s_\nu^{[k]} - s_{\nu-1}^{[k]} = \ell_k(\nu) s_\nu^{[k-1]}$, and summing by parts again, we deduce that

$$\sum_{v=0}^{\infty} h_v^{[k-1]} s_v^{[k-1]} = \sum_{v=0}^{\infty} \frac{h_v^{[k-1]}}{\ell_k(v)} (s_v^{[k]} - s_{v-1}^{[k]}) = -\sum_{v=0}^{\infty} h_v^{[k]} s_v^{[k]}.$$

Thus, (4.10) is proved by induction.

In this proof only, we will denote by $\mathcal{R}_t$, $t \in \mathbb{R}$, the set of all rational functions such that the degree of the denominator is at least t more than the degree of the numerator. We note that each $\mathcal{R}_t$ is a linear space, $\mathcal{R}_u \subseteq \mathcal{R}_t$ for $u \geqslant t$. Moreover, if $R \in \mathcal{R}_t$, and L is a polynomial of precise degree 1, then $R(\cdot+1) - R(\cdot) \in \mathcal{R}_{t+1}$, and $R/L \in \mathcal{R}_{t+1}$. In order to prove (4.11), we will prove by induction that for $k = 1, 2, \ldots$, there exist $R_{k,m} \in \mathcal{R}_{k+m-1}$, $m = 0, \ldots, k-1$, such that

$$h_v^{[k]} = \sum_{m=0}^{k-1} R_{k,m}(v+1) \varDelta^{k-m} h_v, \qquad v = 0, 1, \ldots . \tag{4.12}$$

We will write $R_{k,m} = 0$ if $m < 0$ or $m > k-1$. Eq. (4.12) is obvious if $k = 1$. Suppose $k \geqslant 2$ and (4.12) is proved for $k-1$ in place of k. From definition (4.8), we see that

$$h_v^{[k]} = \frac{1}{\ell_k(v+1)} \left\{ \varDelta h_v^{[k-1]} - \frac{1}{(\alpha+k-1)\ell_k(v)} h_v^{[k-1]} \right\}. \tag{4.13}$$

From the induction hypothesis,

$$\begin{aligned}
\varDelta h_v^{[k-1]} &= \sum_{m=0}^{k-2} \varDelta \left(R_{k-1,m}(v+1) \varDelta^{k-1-m} h_v \right) \\
&= \sum_{m=0}^{k-2} R_{k-1,m}(v+2) \varDelta^{k-m} h_v \\
&\quad + \sum_{m=0}^{k-2} \varDelta^{k-1-m} h_v \left(R_{k-1,m}(v+2) - R_{k-1,m}(v+1) \right) \\
&= \sum_{m=0}^{k-1} \varDelta^{k-m} h_v \left(R_{k-1,m}(v+2) + R_{k-1,m-1}(v+2) - R_{k-1,m-1}(v+1) \right).
\end{aligned} \tag{4.14}$$

The induction hypothesis also gives

$$h_v^{[k-1]} = \sum_{m=0}^{k-2} R_{k-1,m}(v+1) \varDelta^{k-1-m} h_v = \sum_{m=0}^{k-1} R_{k-1,m-1}(v+1) \varDelta^{k-m} h_v. \tag{4.15}$$

We now put for $m = 0, \ldots, k-1$, and $x \in \mathbb{R}$,

$$R_{k,m}(x) := \frac{1}{\ell_k(x+1)} \Bigg(R_{k-1,m}(x+2) + R_{k-1,m-1}(x+2) - R_{k-1,m-1}(x+1) - \frac{1}{(\alpha+k-1)\ell_k(x)} R_{k-1,m-1}(x+1) \Bigg). \tag{4.16}$$

Since $R_{k-1,m-1}(\cdot+2) - R_{k-1,m-1}(\cdot+1) \in \mathcal{R}_{k+m-2}$, it is easy to deduce that $R_{k,m} \in \mathcal{R}_{k+m-1}$. Eqs. (4.13)–(4.15) now imply that (4.12) holds for k. Thus, the proof is complete by induction.

Estimate (4.11) is immediately clear from (4.12). □

Next, we introduce some kernel functions. For $x, y \in \mathbb{R}$, $n = 0, 1, \ldots$, let

$$\begin{aligned} K_{n,1}(x, y) &:= K_{n,1}(\alpha, \beta; x, y) := \sum_{v=0}^{n} {\kappa_v{}^{(\alpha,\beta)}}^{-1} P_v{}^{(\alpha,\beta)}(x) P_v{}^{(\alpha,\beta)}(y), \\ K_{n,k}(x, y) &:= K_{n,k}(\alpha, \beta; x, y) := \sum_{v=0}^{n} \ell_k(v) K_{v,k-1}(x, y), \quad k = 2, 3, \ldots \end{aligned} \tag{4.17}$$

with the convention as usual that $K_{t,k}(x, y) = 0$ if $t < 0$. Applying (4.6) repeatedly, we obtain

$$K_{n,k}(x, 1) = K_{n,k}(1, x) = \frac{\Gamma(n+\alpha+\beta+k+1)}{2^{\alpha+\beta+k}\Gamma(n+\beta+1)\Gamma(\alpha+k)} P_n{}^{(\alpha+k,\beta)}(x). \tag{4.18}$$

We will also use heavily the following product formula proved by Koornwinder [7].

Proposition 4.1. *Let* $\alpha \geqslant \beta \geqslant -1/2$, $\mathcal{R} := [0,1] \times [0,\pi]$, *and for* $x, y \in [-1,1]$, $r \in [0,1]$, $\phi \in [0,\pi]$, *let*

$$F(x, y; r, \phi) := \frac{(1+x)(1+y)}{2} + \frac{(1-x)(1-y)}{2} r^2 + \sqrt{1-x^2}\sqrt{1-y^2}\, r\cos\phi - 1. \tag{4.19}$$

There exists a probability measure $\rho = \rho_{\alpha,\beta}$ *on* $\mathcal{R}$ *such that for* $n = 0, 1, \ldots$, *and* $x, y \in [-1,1]$,

$$P_n{}^{(\alpha,\beta)}(x) P_n{}^{(\alpha,\beta)}(y) = \int_{\mathcal{R}} P_n{}^{(\alpha,\beta)}(1) P_n{}^{(\alpha,\beta)}(F(x, y; r, \phi))\, d\rho(r, \phi). \tag{4.20}$$

Lemma 4.4 and Proposition 4.1 immediately lead to the following lemma, giving (in particular) alternative expressions for the kernel $\Phi_n(H)$.

Lemma 4.5. *Let* $\mathbf{h} := \{h_\nu\}$ *be a sequence with* $h_\nu = 0$ *if* ν *is greater than some positive integer,* $\alpha, \beta \geqslant -1/2$. *Then for* $k = 1, 2, \ldots$, *we have*

$$\Psi(\mathbf{h}, x, y) := \Psi(\alpha, \beta; \mathbf{h}, x, y) := \sum_{\nu=0}^{\infty} h_\nu {\kappa_\nu^{(\alpha,\beta)}}^{-1} P_\nu^{(\alpha,\beta)}(x) P_\nu^{(\alpha,\beta)}(y)$$
$$= (-1)^k \sum_{\nu=0}^{\infty} h_\nu^{[k]} K_{\nu,k}(x, y). \tag{4.21}$$

In particular, for $k = 1, 2, \ldots$,

$$\Psi(\mathbf{h}, x, 1) = \Psi(\mathbf{h}, 1, x)$$
$$= (-1)^k \sum_{\nu=0}^{\infty} h_\nu^{[k]} \frac{\Gamma(\nu + \alpha + \beta + k + 1)}{2^{\alpha+\beta+k}\Gamma(\nu + \beta + 1)\Gamma(\alpha + k)} P_\nu^{(\alpha+k,\beta)}(x). \tag{4.22}$$

If $\alpha \geqslant \beta \geqslant -1/2$, *then for* $k = 1, 2, \ldots$,

$$\Psi(\mathbf{h}, x, y)$$
$$= (-1)^k \int_{\mathcal{R}} \sum_{\nu=0}^{\infty} h_\nu^{[k]} \frac{\Gamma(\nu + \alpha + \beta + k + 1)}{2^{\alpha+\beta+k}\Gamma(\nu + \beta + 1)\Gamma(\alpha + k)}$$
$$\times P_\nu^{(\alpha+k,\beta)}(F(x, y; r, \phi))\, d\rho(r, \phi). \tag{4.23}$$

Proof. We apply Lemma 4.4 with $a_\nu = {\kappa_\nu^{(\alpha,\beta)}}^{-1} P_\nu^{(\alpha,\beta)}(x) P_\nu^{(\alpha,\beta)}(y)$ to obtain (4.21). The equation (4.22) follows from (4.21) and (4.18). In view of Proposition 4.1,

$$a_\nu = {\kappa_\nu^{(\alpha,\beta)}}^{-1} \int_{\mathcal{R}} P_\nu^{(\alpha,\beta)}(1) P_\nu^{(\alpha,\beta)}(F(x, y; r, \phi))\, d\rho(r, \phi).$$

Therefore, (4.23) follows from (4.22). □

Our next lemma gives a general bound on the norms of the kernels in (4.23).

Lemma 4.6. *Let* $\mathbf{h} = \{h_\nu\}$ *be a sequence with* $h_\nu = 0$ if ν *is greater than some positive integer,* $\alpha, \beta \geqslant -1/2$, *and* $K > \max(\alpha, \beta) + 3/2$ *be an integer. Then*

$$\sup_{x \in [-1,1]} \|\Psi(\mathbf{h}, x, \cdot)\|_{\alpha,\beta;1} \leqslant c \|\Psi(\mathbf{h}, 1, \cdot)\|_{\alpha,\beta;1} \leqslant c \sum_{j=1}^{K} \sum_{\nu=0}^{\infty} (\nu + 1)^{j-1} |\Delta^j h_\nu|. \tag{4.24}$$

Proof. First, let $\alpha \geqslant \beta$. In this proof only, let f denote the function $\Psi(\mathbf{h}, 1, y)$. Then using the notation of Askey and Wainger [2, Formula (A-2)], we see that for $x, y \in [-1, 1]$, $\Psi(\mathbf{h}, x, y)$ is equal to the generalized translation $f(y; x)$. Hence, the first inequality in (4.24) follows from [2, Theorem 1]. Since $K > \alpha + 3/2$, we have from [19, Formula (7.34.1)] that for $v = 0, 1, \ldots,$

$$\int_0^1 (1-y)^\alpha |P_v^{(\alpha+K,\beta)}(y)|dy \leqslant c(v+1)^{K-\alpha-2},$$
$$\int_{-1}^0 (1+y)^\beta |P_v^{(\alpha+K,\beta)}(y)|dy \leqslant c(v+1)^{-1/2}.$$

Consequently, for $v = 0, 1, \ldots,$

$$\frac{\Gamma(v+\alpha+\beta+K+1)}{2^{\alpha+\beta+K}\Gamma(v+\beta+1)\Gamma(\alpha+K)}\|P_v^{(\alpha+K,\beta)}\|_{\alpha,\beta;1} \leqslant c(v+1)^{2K-2}.$$

In view of (4.11), we deduce that

$$\sum_{v=0}^{\infty} |h_v^{[K]}| \frac{\Gamma(v+\alpha+\beta+K+1)}{2^{\alpha+\beta+K}\Gamma(v+\beta+1)\Gamma(\alpha+K)}\|P_v^{(\alpha+K,\beta)}\|_{\alpha,\beta;1}$$
$$\leqslant c \sum_{m=0}^{K-1}\sum_{v=0}^{\infty}(v+1)^{K-m-1}|\Delta^{K-m}h_v|.$$

The second estimate in (4.24) now follows from (4.22).

If $\beta > \alpha$, we observe that $\Psi(\alpha, \beta; \mathbf{h}, x, y) = \Psi(\beta, \alpha; \mathbf{h}, -x, -y)$, and

$$\|\Psi(\alpha, \beta; \mathbf{h}, x, \cdot)\|_{\alpha,\beta;1} = \|\Psi(\beta, \alpha; \mathbf{h}, -x, \cdot)\|_{\beta,\alpha;1}. \qquad \square$$

We have now finished with our preparation for the proof of (2.10). We pause in our proof of Theorem 3.1, and indicate how Lemma 4.6 leads to a proof of Theorem 3.2.

Proof of Theorem 3.2 for integer $k > \max(\alpha, \beta) + 1/2$**.** We let

$$h_{v,n} = \begin{cases} \binom{n-v+k}{k} & \text{if } v = 0, \ldots, n, \\ 0 & \text{otherwise} \end{cases}$$

and observe (by induction on j) that with differences applied to the variable v,

$$\Delta^j h_{v,n} = (-1)^j \binom{n-v+k-j}{k-j}, \qquad v = 0, \ldots, n,\ \ j = 1, \ldots, k$$

and

$$\Delta^{k+1} h_{v,n} = \begin{cases} (-1)^{k+1} & \text{if } v = n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{v=0}^{n}\sum_{j=1}^{k+1}(v+1)^{j-1}|\Delta^j h_{v,n}|$$

$$= (n+1)^k + \sum_{v=0}^{n} \sum_{j=1}^{k} (v+1)^{j-1} \binom{n-v+k-j}{k-j}$$

$$\leqslant (n+1)^k + c \sum_{v=0}^{n} \sum_{j=0}^{k-1} \binom{k-1}{j} (v+1)^j (n-v+1)^{k-1-j} \leqslant cn^k.$$

Thus, Lemma 4.6 (applied with $k+1$ in place of K) implies (3.5). □

We now resume our proof of Theorem 3.1, with the preparations for the proof of (2.11). Again, the details are encoded in a number of lemmas. First, we recall some properties of the Jacobi polynomials.

Proposition 4.2. *Let* $\alpha, \beta \geqslant -1/2$. *For* $v = 0, 1, \ldots,$ *we have*

$$|P_v^{(\alpha,\beta)}(x)| \leqslant c \begin{cases} \min((v+1)^{\alpha}, (1-x)^{-\alpha/2-1/4}(v+1)^{-1/2}) & if\ 0 \leqslant x \leqslant 1, \\ \min((v+1)^{\beta}, (1+x)^{-\beta/2-1/4}(v+1)^{-1/2}) & if\ -1 \leqslant x \leqslant 0. \end{cases} \tag{4.25}$$

Further, for any integer $q \geqslant 1$, *there exist complex valued functions* A_m, *continuous on* $(0, \pi)$, *such that for* $\theta \in (0, \pi)$,

$$P_v^{(\alpha,\beta)}(\cos\theta) = 2\mathcal{R}\left\{ \sum_{m=0}^{q-1} A_m(\theta)(v+1)^{-m-1/2} \exp(iv\theta) \right\} + \mathcal{O}((v+1)^{-q-1/2}), \tag{4.26}$$

where the $\mathcal{O}$ *term is uniform on compact subintervals of* $(0, \pi)$.

Proof. Estimate (4.25) is proved in [19, Theorem 7.32.2], the asymptotics (4.26) is proved in [19, Theorem 8.21.9]. □

We start with the case when $x \in (-1, 1)$ and $|y|$ is away from 1.

Lemma 4.7. *Let* $\alpha, \beta \geqslant -1/2$, $x \in (-1, 1)$, $1 - |y| \geqslant \min(1/36, (1-|x|)^2/25)$, $q \geqslant 1$ *be an integer. Let* $h_v = 0$ *if* v *is greater than some integer, and* $\Delta h_v = 0$ if $v \leqslant q+1$. *Then*

$$|\Psi(\mathbf{h}, x, y)| \leqslant \frac{c(q, x)}{|x-y|^q} \sum_{s=0}^{q-1} \sum_{v=0}^{\infty} |\Delta^{q-s} h_v| (v+1)^{-s}, \tag{4.27}$$

where $c(q, x)$ *is bounded for* x *in compact subintervals of* $(-1, 1)$.

Proof. In this proof only, let $x = \cos\theta$, $y = \cos\phi$. Using Lemma 4.5, we obtain

$$\Psi(\mathbf{h}, x, y) = -\sum_{v=0}^{\infty} \Delta h_v K_{v,1}(x, y). \tag{4.28}$$

According to [19, Formula (4.5.2)],

$$\begin{aligned} K_{v,1}&(\cos\theta, \cos\phi)\\ &= \frac{2^{-\alpha-\beta}}{2v+\alpha+\beta+2}\frac{\Gamma(v+2)\Gamma(v+\alpha+\beta+2)}{\Gamma(v+\alpha+1)\Gamma(v+\beta+1)}\\ &\quad\times\frac{P_{v+1}{}^{(\alpha,\beta)}(\cos\theta)P_v{}^{(\alpha,\beta)}(\cos\phi) - P_v{}^{(\alpha,\beta)}(\cos\theta)P_{v+1}{}^{(\alpha,\beta)}(\cos\phi)}{\cos\theta-\cos\phi}. \end{aligned} \tag{4.29}$$

Writing $B_{1,m,\ell}(\theta,\phi) = A_m(\theta)A_\ell(\phi)e^{i\theta}$, and $B_{2,m,\ell}(\theta,\phi) = A_m(\theta)\overline{A_\ell(\phi)}e^{i\theta}$, and using (4.26), we deduce that

$$\begin{aligned} P_{v+1}&{}^{(\alpha,\beta)}(\cos\theta)P_v{}^{(\alpha,\beta)}(\cos\phi)\\ &= 2\Re\left\{\sum_{m,\ell=0}^{q-1} B_{1,m,\ell}(\theta,\phi)(v+1)^{-m-\ell-1}\exp(iv(\theta+\phi))\right\}\\ &\quad+2\Re\left\{\sum_{m,\ell=0}^{q-1} B_{2,m,\ell}(\theta,\phi)(v+1)^{-m-\ell-1}\exp(iv(\theta-\phi))\right\} + \mathcal{O}((v+1)^{-q-1}). \end{aligned} \tag{4.30}$$

We interchange the roles of θ and ϕ above, and substitute the resulting asymptotics back in (4.29) to obtain (with $B_{3,m,\ell}(\theta,\phi) = B_{1,m,\ell}(\theta,\phi) - B_{1,m,\ell}(\phi,\theta)$ and $B_{4,m,\ell}(\theta,\phi) = B_{2,m,\ell}(\theta,\phi) - \overline{B_{2,m,\ell}(\phi,\theta)}$)

$$\begin{aligned} 2^{\alpha+\beta}(x-y)K_{v,1}(x,y) &= \Re\left\{\sum_{m,\ell=0}^{q-1} B_{3,m,\ell}(\theta,\phi)(v+1)^{-m-\ell}\exp(iv(\theta+\phi))\right\}\\ &\quad+\Re\left\{\sum_{m,\ell=0}^{q} B_{4,m,\ell}(\theta,\phi)(v+1)^{-m-\ell}\exp(iv(\theta-\phi))\right\}\\ &\quad+\mathcal{O}((v+1)^{-q}). \end{aligned} \tag{4.31}$$

Now, we write $g_v = \varDelta h_v$ if $v \geqslant 0$ and $g_v = 0$ if $v < 0$. We recall from [14, Proposition 2.2] that for $\psi \in \mathbb{R}$, and $r \in \mathbb{R}$,

$$\left|\sum_{v\in\mathbb{Z}} g_v(v+1)^{-r}e^{iv\psi}\right| \leqslant \frac{c}{|\psi \bmod 2\pi|^{q-1}}\sum_{v\in\mathbb{Z}}\left|\varDelta^{q-1}\left(g_v(v+1)^{-r}\right)\right|.$$

Using the Leibniz formula for differences, and recalling that $g_j = \varDelta h_j = 0$ if $j \leqslant q+1$, we deduce that

$$\begin{aligned} \left|\sum_{v\in\mathbb{Z}} g_v(v+1)^{-r}e^{iv\psi}\right| &\leqslant \frac{c}{|\psi \bmod 2\pi|^{q-1}}\sum_{s=0}^{q-1}\sum_{v\in\mathbb{Z}}|\varDelta^{q-1-s}g_v|(v+1)^{-r-s}\\ &\leqslant \frac{c}{|\psi \bmod 2\pi|^{q-1}}\sum_{s=0}^{q-1}\sum_{v=0}^{\infty}|\varDelta^{q-s}h_v|(v+1)^{-r-s}. \end{aligned} \tag{4.32}$$

From (4.28), (4.31), (4.32), we conclude that

$$2^{\alpha+\beta}|x-y||\Psi(\mathbf{h},x,y)| \leqslant \sum_{m,\ell=0}^{q-1}\frac{|B_{3,m,\ell}(\theta,\phi)|}{|\theta+\phi|^{q-1}}\sum_{s=0}^{q-1}\sum_{v=0}^{\infty}|\Delta^{q-s}h_v|(v+1)^{-m-\ell-s}$$
$$\times\sum_{m,\ell=0}^{q-1}\frac{|B_{4,m,\ell}(\theta,\phi)|}{|\theta-\phi|^{q-1}}\sum_{s=0}^{q-1}\sum_{v=0}^{\infty}|\Delta^{q-s}h_v|(v+1)^{-m-\ell-s}$$
$$+c\sum_{v=0}^{\infty}(\Delta h_v)(v+1)^{-q}. \tag{4.33}$$

Since $x\in(-1,1)$, and $|y|\leqslant c(x)$, $2\pi-c_1(x)\geqslant\theta+\phi\geqslant c_2(x)$. Also,

$$|\theta-\phi|\geqslant c\left|\sin\left((\theta-\phi)/2\right)\right|\geqslant c(x)|x-y|.$$

Finally, $|x-y|\geqslant c|x-y|^q$. Hence, (4.33) implies (4.27). $\square$

In the next lemma, we estimate the quantity $P_v{}^{(\alpha,\beta)}(F(x,y;r,\phi))$ (cf. (4.19)) in the case when $x\in(-1,1)$ and $|y|$ is close to 1. We will use it with $\alpha+K$ in place of α for some integer K.

Lemma 4.8. *Let* $\alpha\geqslant\beta\geqslant-1/2$, $x\in(-1,1)$, $0\leqslant 1-|y|\leqslant\min(1/36,(1-|x|)^2/25)$. *Then for* $v=0,1,\ldots,$

$$|P_v{}^{(\alpha,\beta)}(F(x,y;r,\phi))|\leqslant c(x)\begin{cases}(v+1)^{-1/2} & \text{if } 0\leqslant y\leqslant 1,\\ (v+1)^{\beta} & \text{if } -1\leqslant y<0,\end{cases} \tag{4.34}$$

where $c(x)$ *is bounded on compact subintervals of* $(-1,1)$.

Proof. It is not difficult to calculate that for $r\in[0,1]$, $\phi\in[0,\pi]$,

$$F(x,y;r,\phi)=x+\varepsilon_1(x,y;r,\phi)=(1-x)r^2-1+\varepsilon_2(x,y;r,\phi), \tag{4.35}$$

where, in this proof only,

$$\varepsilon_1(x,y;r,\phi):=\frac{(1-x)(1-y)r^2}{2}-\frac{(1+x)(1-y)}{2}$$
$$+\sqrt{1-x^2}\sqrt{1-y^2}r\cos\phi \tag{4.36}$$

and

$$\varepsilon_2(x,y;r,\phi):=\frac{(1+x)(1+y)}{2}-\frac{(1-x)(1+y)r^2}{2}$$
$$+\sqrt{1-x^2}\sqrt{1-y^2}r\cos\phi. \tag{4.37}$$

Both of the functions ε_1, ε_2 can be estimated in the same way. For $r \in [0,1]$, $\phi \in [0,\pi]$, $x \in (-1,1)$, and $y \in [0,1]$, we have

$$\begin{aligned}|\varepsilon_1(x,y;r,\phi)| &\leqslant \frac{(1+x)(1-|y|)}{2} + \frac{(1-x)(1-|y|)}{2} + \sqrt{1-y^2}\\ &\leqslant 1-|y| + \sqrt{2}\sqrt{1-|y|} \leqslant (5/2)\sqrt{1-|y|}.\end{aligned} \tag{4.38}$$

Similarly, for $r \in [0,1]$, $\phi \in [0,\pi]$, $x \in (-1,1)$, and $y \in [-1,0]$, we have

$$\begin{aligned}|\varepsilon_2(x,y;r,\phi)| &\leqslant \frac{(1+x)(1-|y|)}{2} + \frac{(1-x)(1-|y|)}{2} + \sqrt{1-y^2}\\ &\leqslant 1-|y| + \sqrt{2}\sqrt{1-|y|} \leqslant (5/2)\sqrt{1-|y|}.\end{aligned} \tag{4.39}$$

Now, let $y \geqslant 0$. If $|x| \leqslant (5/2)\sqrt{1-|y|}$, then the first equation in (4.35) shows that $|F(x,y;r,\phi)| \leqslant 5\sqrt{1-|y|} \leqslant 5/6$. Consequently, (4.25) leads to the first estimate in (4.34) when $|x| \leqslant (5/2)\sqrt{1-|y|}$. If $|x| > (5/2)\sqrt{1-|y|}$, then $F(x,y;r,\phi)$ has the same sign as x. If $x > (5/2)\sqrt{1-|y|}$, then the fact that

$$1 - F(x,y;r,\phi) \geqslant 1 - x - (5/2)\sqrt{1-|y|} \geqslant (1-x)/2,$$

along with (4.25), leads to the first estimate in (4.34) again. If $x < -(5/2)\sqrt{1-|y|}$ then $\sqrt{1-|y|} \leqslant (1-|x|)/5 = (1+x)/5$, and

$$1 + F(x,y;r,\phi) = 1 + x + \varepsilon_1(x,y;r,\phi) \geqslant 1 + x - (5/2)\sqrt{1-|y|} \geqslant (1+x)/2.$$

Therefore, (4.25) leads to the first estimate in (4.34) in this final case as well.

Next, let $y \leqslant 0$. We will use the second equation in (4.35), and bound (4.39). If $F(x,y;r,\phi) > 0$ then for $r \in [0,1]$, $\phi \in [0,\pi]$,

$$\begin{aligned}1 - F(x,y;r,\phi) &= 2 - (1-x)r^2 - \varepsilon_2(x,y;r,\phi)\\ &= 1 + x + (1-x)(1-r^2) - \varepsilon_2(x,y;r,\phi)\\ &\geqslant 1 + x - \varepsilon_2(x,y;r,\phi).\end{aligned}$$

Since

$$|\varepsilon_2(x,y;r,\phi)| \leqslant (5/2)\sqrt{1-|y|} \leqslant (1-|x|)/2 \leqslant (1+x)/2,$$

we deduce that $1 - F(x,y;r,\phi) \geqslant (1+x)/2$. Therefore, (4.25) leads to

$$|P_\nu{}^{(\alpha,\beta)}(F(x,y;r,\phi))| \leqslant c(x)(\nu+1)^{-1/2}, \quad \text{if } F(x,y;r,\phi) > 0. \tag{4.40}$$

Since $|P_\nu{}^{(\alpha,\beta)}(F(x,y;r,\phi))| \leqslant c(\nu+1)^\beta$ when $F(x,y;r,\phi) \leqslant 0$ (cf. (4.25)), this completes the proof of the lemma in the case when $y \leqslant 0$ as well. $\square$

The next lemma is the analogue of Lemma 4.7 in the case when $x \in (-1,1)$ and $|y|$ is close to 1.

Lemma 4.9. *Let* $\alpha, \beta \geqslant -1/2$, $K \geqslant 1$ be an integer, $x \in (-1, 1)$ *and* $0 \leqslant 1-|y| \leqslant \min(1/36, (1-|x|)^2/25)$. *Let* $h_\nu = 0$ *for all sufficiently large* ν. *Then*

$$|\Psi(\mathbf{h}, x, y)| \leqslant \frac{c(K, x)}{|x-y|^K} \sum_{m=0}^{K-1} \sum_{\nu=0}^{\infty} |\Delta^{K-m} h_\nu|(\nu+1)^{\alpha+\beta+1-m}, \tag{4.41}$$

where $c(K, x)$ *is bounded on compact subintervals of* $(-1, 1)$.

Proof. First, let $\alpha \geqslant \beta$. In view of (4.23),

$$|\Psi(\mathbf{h}, x, y)| \leqslant c \sum_{\nu=0}^{\infty} |h_\nu^{[K]}|(\nu+1)^{\alpha+K} \int_{\mathcal{R}} |P_\nu{}^{(\alpha+K,\beta)}(F(x, y; r, \phi))| d\rho(r, \phi).$$

Now we use estimate (4.11) for $|h_\nu^{[K]}|$ and (4.34) for $|P_\nu{}^{(\alpha+K,\beta)}(F(x, y; r, \phi))|$, and recall that $|x-y| \geqslant c(x)$ to arrive at (4.41). If $\beta > \alpha$, we note that $\Psi(\alpha, \beta; \mathbf{h}, x, y) = \Psi(\beta, \alpha; \mathbf{h}, -x, -y)$. $\square$

Since $\Psi(\mathbf{h}, x, y) = \Psi(\mathbf{h}, y, x)$, the above lemma also gives the bounds we need in the case of $\Psi(\mathbf{h}, \pm 1, y)$ when y is in a compact subinterval of $(-1, 1)$. In the following last lemma before the proof of Theorem 3.1, we state the bounds for $\Psi(\mathbf{h}, \pm 1, y)$ in a more precise manner than in Lemma 4.9. For the purpose of this paper, the lemma is needed only to cover the case of $\Psi(\mathbf{h}, \pm 1, \mp 1)$. We state it here in the more general form, because its proof is immediate from our work so far in this paper, and because we need it for other applications. In particular, in the important case when $\alpha = \beta = q/2 - 1$ for some integer $q \geqslant 1$, Lemma 4.10 below enables one to obtain bounds on kernels based on spherical polynomials on a Euclidean sphere embedded in $\mathbb{S}^q$ [9,16]. In this case, an anlogue of the following lemma was obtained by Narcowich, Petrushev, and Ward, and was recently announced by Narcowich in a lecture in Oberwolfach (May, 2004) [16] and by Petrushev in a lecture in Nashville (December, 2003). We acknowledge the privilege of being in the audience in both of these lectures, as well as the ensuing discussions with many mathematicians, including Freeden, Narcowich, Prestin, Reimer, Sloan, Ward, and Xu.

Lemma 4.10. *Let* $\alpha, \beta \geqslant -1/2$, $K \geqslant 1$ *be an integer,* $h_\nu = 0$ *for all sufficiently large* ν. *Then*

$$|\Psi(\mathbf{h}, 1, y)| \leqslant c \begin{cases} \displaystyle\sum_{\nu=0}^{\infty} \min\left((\nu+1)^2, \frac{1}{1-y}\right)^{\alpha/2+K/2+1/4} \\ \displaystyle\quad \times \sum_{m=0}^{K-1} (\nu+1)^{\alpha+1/2-m} |\Delta^{K-m} h_\nu| & \text{if } 0 \leqslant y < 1, \\ \displaystyle\sum_{\nu=0}^{\infty} (\nu+1)^{\alpha+\beta+1} \sum_{m=0}^{K-1} (\nu+1)^{-m} |\Delta^{K-m} h_\nu| & \text{if } -1 \leqslant y < 0, \end{cases} \tag{4.42}$$

and

$$\Psi(\mathbf{h}, -1, y)| \leqslant c \begin{cases} \displaystyle\sum_{v=0}^{\infty} \min\left((v+1)^2, \frac{1}{1+y}\right)^{\beta/2+K/2+1/4} & \\ \quad \displaystyle\times \sum_{m=0}^{K-1} (v+1)^{\beta+1/2-m} |\Delta^{K-m} h_v| & \text{if } -1 < y \leqslant 0, \\ \displaystyle\sum_{v=0}^{\infty} (v+1)^{\alpha+\beta+1} \sum_{m=0}^{K-1} (v+1)^{-m} |\Delta^{K-m} h_v| & \text{if } 0 < y \leqslant 1. \end{cases} \tag{4.43}$$

Proof. In view of (4.25),

$$|P_v^{(\alpha+K,\beta)}(y)| \leqslant c \begin{cases} \min((v+1)^{\alpha+K}, (1-y)^{-\alpha/2-K/2-1/4}(v+1)^{-1/2}) & \text{if } 0 \leqslant y < 1, \\ c(v+1)^{\beta} & \text{if } -1 \leqslant y < 0. \end{cases}$$

Therefore, (4.22) and (4.11) lead to (4.42). Estimate (4.43) follows from (4.42) by observing that $\Psi(\alpha, \beta; \mathbf{h}, -1, y) = \Psi(\beta, \alpha; \mathbf{h}, 1, -y)$. □

Finally, we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. The hypothesis on the function h implies that for each $n \geqslant c(Q)$, the sequence $\{h_{v,n}\}$ satisfies all the conditions on the sequence $\mathbf{h}$ in the Lemmas 4.6, 4.7, 4.9, and 4.10. Each of the sums on v in each of these lemmas is for $c(\delta)n \leqslant v \leqslant c_1 n$. Also, the mean value theorem implies that with the differences applied to the variable v and for integer $r \geqslant 1$,

$$\sum_{v=c(\delta)n}^{c_1 n} |\Delta^r h_{v,n}| \leqslant c n^{-r+1} V(h^{(r-1)}),$$

where $V(g)$ denotes the total variation of g. Therefore, for any $s \in \mathbb{R}$, and integer $r \geqslant 1$,

$$\sum_{n=0}^{\infty} (v+1)^s |\Delta^r h_{v,n}| = \sum_{v=c(\delta)n}^{c_1 n} (v+1)^s |\Delta^r h_{v,n}| \leqslant c n^{s-r+1} V(h^{(r-1)}).$$

With these observations, Lemma 4.6 implies that

$$\sup_{n \geqslant 0, x \in \mathcal{S}_\mu} \|\Phi_n(H, x, \cdot)\|_1 < c \sum_{j=0}^{K-1} V(h^{(j)}),$$

which is (2.10). Condition (2.11) follows from Lemmas 4.7 (with $q = \lceil Q \rceil + 1$), 4.9, and 4.10. □

Proof of Theorem 3.3. In this proof only, we will write $p_n := {\kappa_n}^{(\alpha,\beta)^{-1/2}} {P_n}^{(\alpha,\beta)}$, and $w = w_{\alpha,\beta}$. In this proof only, let $g_{k,m} = h_{k,2^m} - h_{k,2^{m-1}}$, $y_{k,n} = h_{k,2^{n+1}} - h_{k,2^{n-2}}$, and ψ_j be defined for $j \in \mathbb{Z}$ by

$$\psi_j(x) = \left(h(2^j x) - h(2^{j+1}x)\right)(h(x/2) - h(4x)).$$

Then $g_{k,m} = 0$ if $k \leqslant 2^{m-2}$ or $k > 2^m$, and $y_{k,n} = 0$ if $k \leqslant 2^{n-3}$ or $k > 2^{n+1}$. Hence, $g_{k,m} y_{k,n} = 0$ if $|n-m| \geqslant 3$. Therefore, for $x \in \mathbb{R}$, (3.6) implies that for $m \geqslant 3$,

$$\begin{aligned}
\tau_m(H, f, x) &= \sum_{n=0}^{\infty} \int d_n(t) \int_{-1}^{1} \sum_{\ell=0}^{\infty} y_{\ell,n} p_\ell(y) p_\ell(t) \sum_{k=0}^{\infty} g_{k,m} p_k(x) p_k(y) w(y) dy dv(t) \\
&= \sum_{n=m-2}^{m+2} \int d_n(t) \sum_{k=0}^{\infty} y_{k,n} g_{k,m} p_k(t) p_k(x) dv_n(t) \\
&= \sum_{j=-2}^{2} \int d_{m+j}(t) \sum_{k=0}^{\infty} y_{k,m+j} g_{k,m} p_k(t) p_k(x) dv_{m+j}(t) \\
&= \sum_{j=-2}^{2} \int d_{m+j}(t) \sum_{k=0}^{\infty} \psi_j(k/2^{m+j}) p_k(t) p_k(x) dv_{m+j}(t). \qquad (4.44)
\end{aligned}$$

Now, we observe that each of the functions $\psi_j(|j| \leqslant 2)$ satisfies the conditions of Theorem 3.1 to ensure that (2.10), (2.14), and (2.11) hold for each of the matrices $M_j = (\psi_j(k/n))$, $|j| \leqslant 2$. Let J be the interval, centered at x_0, and having length $|I|/2$. Then for $x \in J$ and $t \in [-1, 1] \setminus I$,

$$\left| \sum_{k=0}^{\infty} \psi_j(k/2^{m+j}) p_k(x) p_k(t) \right| \leqslant c(I) 2^{-mQ}.$$

Hence, for $j = 0, \pm 1, \pm 2$, and $x \in J$,

$$\begin{aligned}
&\left| \int_{t \in [-1,1] \setminus I} d_{m+j}(t) \sum_{k=0}^{\infty} \psi_j(k/2^{m+j}) p_k(t) p_k(x) dv_{m+j}(t) \right| \\
&\quad \leqslant c(I) 2^{-mQ} \|d_{m+j}\|_{v_{m+j};p} \leqslant c(I) 2^{-mQ}. \qquad (4.45)
\end{aligned}$$

Therefore, denoting by $\chi(t)$ the characteristic function of I, we obtain that for $x \in J$ and $j = 0, \pm 1, \pm 2$,

$$\begin{aligned}
&\left| \int d_{m+j}(t) \sum_{k=0}^{\infty} \psi_j(k/2^{m+j}) p_k(t) p_k(x) dv_{m+j}(t) \right| \\
&\quad \leqslant \left| \int d_{m+j}(t) \chi(t) \sum_{k=0}^{\infty} \psi_j(k/2^{m+j}) p_k(t) p_k(x) dv_{m+j}(t) \right| + \frac{c(I)}{2^{mQ}} \qquad (4.46)
\end{aligned}$$

Using (2.10), (2.14), and Lemma 4.1 with $m_2 = \mu_{\alpha,\beta}$ and $m_1 = v_{m+j}$, we obtain that

$$\left\| \int d_{m+j}(t)\chi(t) \sum_{k=0}^{\infty} \psi_j(k/2^{m+j}) p_k(t) p_k(\cdot) dv_{m+j}(t) \right\|_p \leqslant c\|d_{m+j}\chi\|_{v_{m+j};p} = c\|d_{m+j}\|_{v_{m+j};I,p}. \tag{4.47}$$

Along with (4.44), this implies that

$$\|\tau_m(H, f)\|_{J,p} \leqslant c(I) \left\{ \sum_{j=-2}^{2} \|d_{m+j}\|_{v_{m+j};I,p} + 2^{-mQ} \right\}.$$

Therefore, $\{\|\tau_m(H, f)\|_{J,p}\} \in \mathsf{b}_{\rho,\gamma}$, and Theorem 2.1 implies that $f \in B_{p,\rho,\gamma}(x_0)$. $\square$

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